Linear Algebra
[KOMS120301] - 2023/2024

12.1 - Basis and Dimension

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Basis of vector space

Intuitive example

In $\mathbb{R}^3 \rightarrow$ Let $\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$

Every vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$ can be expressed as a linear combination of the vector basis, namely:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

In $\mathbb{R}^n \rightarrow$ This can be generalized for the Euclidean vector space $\mathbb{R}^n$

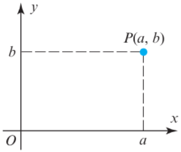
Let: $\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \mathbf{e}_3 = (0, 0, 0, \ldots, 1)$

Every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ can be expressed as:

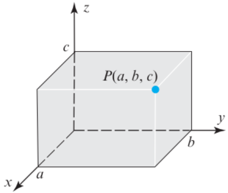
$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{v}_2 + \cdots + v_n\mathbf{v}_n$$

Can a vector space have more than one basis?
What about the basis of general vector space V?

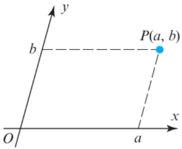
Rectangular and non-rectangular linear system



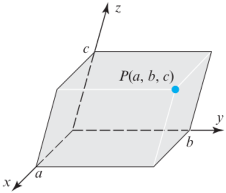
Coordinates of P in a rectangular coordinate system in 2-space.



Coordinates of P in a rectangular coordinate system in 3-space.



Coordinates of P in a nonrectangular coordinate system in 2-space.



Coordinates of P in a nonrectangular coordinate system in 3-space.

In linear algebra, coordinate systems are commonly specified using **vectors** *rather than coordinate axes.*

Formal definition of basis

If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a basis for V if the following two conditions hold:

1. S is linearly independent;
2. S spans V.

Example 1: standard basis for $\mathbb{R}^n$

The standard basis for $\mathbb{R}^n$ is the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

This means that: $\forall \mathbf{v} \in V$, then $\exists k_1, k_2, \ldots, k_n \in \mathbb{R}$, s.t.:

$$\mathbf{v} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n$$

Example (specific case, in $\mathbb{R}^3$)

In $\mathbb{R}^3$, we have the standard basis:

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

Example 2: standard basis for P_n

Show that the set $S = \{1, x, x^2, \ldots, x^n\}$ is a standard basis for vector space P_n of polynomials.

Solution:
By the theorem, it should be showed that the polynomials in S are linearly independent, and span P_n.

Denote the polynomials by vectors:

$$\mathbf{p}_0 = 1,\ \mathbf{p}_1 = x,\ \mathbf{p}_2 = x^2,\ \ldots,\ \mathbf{p}_n = x^n$$

We showed (in the previous discussion) that the vectors span P_n, and they are linearly independent.

Example 2: another basis for $\mathbb{R}^3$

Show that the vectors:

$$\mathbf{v}_1 = (1, 2, 1), \ \mathbf{v}_2 = (2, 9, 0), \ \text{and } \mathbf{v}_3 = (3, 3, 4)$$

form a basis for $\mathbb{R}^3$.

Solution:
It must be showed that the vectors are linearly independent and span $\mathbb{R}^3$.

- *Linear independence*: the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_3\mathbf{v}_3 = \mathbf{0}$$

  has only the trivial solution.

- *Span the vector space $\mathbb{R}^3$*: every vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ can be expressed as:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_3\mathbf{v}_3 = \mathbf{b}$$

Example 2 (*cont.*)

The vector equations can be expressed as linear systems:

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 \qquad\quad + 4c_3 = 0 \end{cases} \qquad \begin{cases} c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 \qquad\quad + 4c_3 = b_3 \end{cases}$$

To show that the homogeneous linear system (*left*) has only trivial solution and the system (*right*) has a unique solution, is equivalent to showing that the coefficient matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

has nonzero determinant.

Task: Prove that $\det(A) \neq 0$.

Uniqueness of basis representation

Theorem (Uniqueness)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the following form, in exactly one way.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

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$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Proof.

Suppose that $\mathbf{v}$ can be expressed in another linear combination, say:

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Substracting two equations gives:

$$\underline{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

Since vectors in S are linearly independent, then:

$$c_1 - k_1 = 0,\ c_2 - k_2 = 0,\ \dots,\ c_n - k_n = 0$$

meaning that: $c_1 = k_1,\ c_2 = k_2,\ \dots,\ c_n = k_n$

Dimension

The number of vectors in a basis

A vector space may have more than one basis which are of the same size.

Theorem (Size of Basis)

All bases for a finite-dimensional vector space have the same number of vectors.

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Theorem

Let V be an n-dimensional vector space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be any basis.

1. *If a set in V has more than n vectors, then it is linearly dependent.*
2. *If a set in V has fewer than n vectors, then it does not span V.*

Proof.

The statements follow because the vectors in S are linearly independent. $\square$

Dimension

The dimension of a finite-dimensional vector space V is defined to be the number of vectors in a basis for V.

The zero vector space is defined to have dimension zero.

Example (Dimensions of some familiar vector spaces)

$$\begin{aligned} \dim(\mathbb{R}^n) &= n && \text{[the standard basis has } n \text{ vectors]} \\ \dim(P_n) &= n+1 && \text{[the standard basis has } n+1 \text{ vectors]} \\ \dim(M_{mn}) &= mn && \text{[the standard basis has } mn \text{ vectors]} \end{aligned}$$

Task: *What is the standard basis for each of the vector space?*

Example 1: dimension of *span*(S)

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the set of linearly independent vectors.

Prove that $\dim(span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}) = n$.

Solution:

Every vector in $span(S)$ can be expressed as a linear combination of the vectors in S.

Hence, S is the basis of $span(S)$.

By the "Size of Basis" theorem,

$$\dim(span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}) = n$$

Example 2: dimension of a solution space

Given the following linear system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_5 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0 \end{cases}$$

Find the dimension of the solution space of the linear system.

Solution:

- Find the solution of the system:

$$x_1 = -3r - 4s - 2t,\; x_2 = r,\; x_3 = -2s,\; x_4 = s,\; x_5 = t,\; x_6 = 0$$

- In vector form:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) &= (-3r - 4s - 2t, r, -2s, s, t, 0) \\ &= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)\end{aligned}$$

Example 2 (*cont.*)

- So the following vectors span the vector space:

  $\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0)$, $\mathbf{v}_2 = (-4, 0, -2, 1, 0, 0)$, $\mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$

- Check that the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. It should be showed that the vector equation:

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

  has only trivial solution, i.e. $c_1 = 0, c_2 = 0, c_3 = 0$.

  Verify it!

- If it is, then S is a basis of the solution space, and $\dim(S) = 3$.

Dimension of subspace

Theorem

If W is a subspace of a finite-dimensional vector space V, then:

1. *W is finite-dimensional;*
2. $\dim(W) \leq \dim(V)$*;*
3. *$W = V$ if and only if* $\dim(W) = \dim(V)$*.*

Proof.

See page 225 of "Elementary Linear Algebra Applications Version (Howard Anton, Chris Rorres - Edisi 1 - 2013)". $\square$

to be continued...